NAME (print): KEY

Math 203 Spring 2013—Exam 3

Instructor: Shapiro

Work carefully and neatly and remember that I cannot grade what I cannot read. You must show all relevant work in the appropriate space. You may receive no credit for a correct answer if there is insufficient supporting work. Notes, books and graphing or programable calculators are NOT ALLOWED.

[18pt] 1. Fill in the blanks with *A(lways), S(ometimes), N(ever)* so that the following are correct statements.

(a) If dim $V = n$ and if S is a linearly independent subset of V with n vectors, then S __A__ spans V.

(b) If A is a 4×7 matrix, then the dimension of Nul A is __S__ equal to three.

(c) The vector $3\mathbf{u} + 2\mathbf{v}$ is __A__ in Span $\{\mathbf{u}, \mathbf{v}\}$.

(d) If A is a non-invertible square matrix, then the columns of A are __N__ linearly independent.

(e) If A is invertible, then Nul A __A__ consists of only the zero vector.

(f) If dim $V = 6$ and if S is a subset of V with 7 vectors, then S __S__ spans V.

[8pts] 2. Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

Show that H is not a subspace of R^2.

Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

Then both u and v are in H.

However, $u+v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is <u>not</u> in H

3. $A = \begin{array}{c} \begin{matrix} v_1 & \quad v_2 & \quad v_3 & \quad v_4 & \quad v_5 \end{matrix} \\ \begin{bmatrix} 24 & -12 & 12 & -72 & 96 \\ 5 & -10 & -20 & 15 & -10 \\ -49 & 56 & 70 & 21 & -70 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \end{array}$ and $B = \begin{bmatrix} 1 & 0 & 6 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

[9pt] (a) What is the dimension of: (i) nul A __3__; (ii) col A __2__; (iii) row A __2__?

[12pt] (b) Give a basis for each of (i) nul A; (ii) col A; (iii) row A.

$$\left.\begin{aligned} x_1 &= -6x_3 + 5x_4 - 6x_5 \\ x_2 &= -3x_3 + 4x_4 - 4x_5 \end{aligned}\right\} \text{3 free variables}$$

$$\begin{pmatrix} -6 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

ii) $\{v_1, v_2\}$

iii)

$(1\ 0\ 6\ -5\ 6)$

$(0\ 1\ 3\ -4\ 4)$

[5pt] (c) Write the fourth column of A as a linear combination of the first three columns, or explain why that cannot be done. (Hint: This is quick)

Look at B

column 4 = -5 col 1 $- 4$ col 3

Hence $\boxed{v_4 = -5v_1 - 4v_2}$ or $v_4 = v_1 - v_2 - v_3$

[16pt] 4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for a vector space V and suppose that $\mathbf{a}_1 = 2\mathbf{b}_1 + 2\mathbf{b}_2$, and $\mathbf{a}_2 = \mathbf{b}_1 + 2\mathbf{b}_2$.

(a) Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{A}$.

$$\underset{B\leftarrow A}{P} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \quad \text{So } \underset{A\leftarrow B}{P} = \underset{B\leftarrow A}{P}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{A}}$ for $\mathbf{x} = 3\mathbf{b}_1 - 1\mathbf{b}_2$

$$\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7 \\ -8 \end{pmatrix} = \boxed{\begin{pmatrix} 7/2 \\ -4 \end{pmatrix}}$$

5. Let $\mathbf{B} = \{1 + 3t + t^2, t + 2t^2, 1 + 3t + t^2, -1 + t\}$.

[8pt] (a) Show that $\mathbf{B}$ spans $\mathbb{P}_2$.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 3 & 0 & 3 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -3 & 0 & 4 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -3 & 0 & 4 \\ 0 & 0 & 0 & 7/3 \end{pmatrix}$$

-3, -1, $1/3$

There's a pivot in each row! so columns span

[8pt] (b) Find a subset of $\mathbf{B}$ that is a basis of $\mathbb{P}_2$.

The vectors corresponding to columns 1, 2 and 4 span and so are a basis. They are

$\{1+3t+t^2, \; t+2t^2, \; -1+t\}$

[16pt] 6. Suppose that in a small town 40% of the people walk to work, while the rest drive to work. Suppose that each year the proportion of people that either walk or drive to work changes according to the stochastic matrix

$$\begin{array}{c|cc} & \text{Walk} & \text{Drive} \\ \hline \text{Walk} & .85 & .2 \\ \text{Drive} & .15 & .8 \end{array}.$$

(a) Next year what percentage of the town will walk to work?

$$\begin{pmatrix} .85 & .2 \\ .15 & .8 \end{pmatrix} \begin{pmatrix} .4 \\ .6 \end{pmatrix} = \begin{pmatrix} .34 + .12 \\ .06 + .48 \end{pmatrix}$$

$$= \begin{pmatrix} .46 \\ .54 \end{pmatrix}$$

$\boxed{\text{46\% will walk next year}}$

(b) In the long run, what percentage of the people will walk to work?

$$\begin{pmatrix} .85 & .2 \\ .15 & .8 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -.15 & .2 \\ .15 & -.2 \end{pmatrix} \sim \begin{pmatrix} -15 & 20 \\ 15 & -20 \end{pmatrix}$$

$$\begin{pmatrix} -15 & 20 \\ 0 & 0 \end{pmatrix} \text{ or } \quad 15x_1 = 20x_2$$

set $x_1 = 1$

$\boxed{\text{Thus 4/7 will walk}}$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \; q = \begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix}$$